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The goal of this paper is to introduce and illustrate a new approach to the stability analysis of sample-paths of nonlinear stochastic economic models with non-stationary components. We place our study within the mathematical theory of random dynamical systems and apply the concept of a random fixed point which is tailor-made for the study of the long-term behavior of sample-paths in stochastic systems. The main tool for the application of this approach is a Banach-type fixed point theorem for non-stationary random dynamical systems which is proved here. The concept and the theorem are thoroughly explained and illustrated by examples from stochastic growth theory.

Keywords: Sample-Path Stability, Random Fixed Points, Non-Stationary Random Dynamical Systems

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1. Introduction

In search of an appropriate concept to describe the long-term behavior and the stability properties of sample-paths of dynamic economic models with randomness, several notions such as "noisy steady states" have been coined in the economic literature in recent years. Most of these approaches, however, are not satisfactory from a mathematical point of view because they lack a concise relation to the dynamics of the stochastic systems considered.

This paper aims at the development of an approach which permits an analysis of the (global) stability properties of sample-paths of stochastic dynamic economic models in a coherent mathematical framework. In particular we want to allow for non-stationary exogenous perturbations of stochastic systems. Our approach is pursued in the theory of random dynamical systems (Arnold [2]) with discrete time. We restrict ourselves to the class of models which are described by a noise-dependent time-one map. This so-called "stochastic law of motion" determines the state of an economy in any period in time given the realization of the state of an exogenous system (a stochastic process) and the state of the economy in the preceding period. To give an idea, many macroeconomic models – in particular models from economic growth theory – possess such a representation, see e.g. the intertemporal models in Azariadis [3], the study of AK and Solow growth models by Binder and Pesaran [4], and Mirman's [12,13] pioneering analysis of neoclassical stochastic growth models. Deterministic models which can possibly be extended to the stochastic case include dynamic models of international trade with externalities in production by Datta and Mirman [6,7] and earlier work by Fischer and Mirman [8]. Models in which non-stationarity enters through the beliefs of economic agents are introduced in Kurz [11].

The framework presented in this paper can for instance be used to study the impact of time-dependent economic policy and the effect of non-stationary exogenous technological change in dynamic economic models. In the examples we consider non-stationary changes of regimes and recurrent poverty due to insufficient productivity. Since our approach deals with general (in particular not necessarily stationary) stochastic perturbations, our analysis also applies to models in which the realization of time-dependent exogenous parameters is given by unfiltered (e.g. not seasonally adjusted) empirical data[1]. However, we have to stress that our approach is formulated for systems with an explicit law of motion. The results in this paper are, for instance, not directly applicable to stochastic optimal control models without closed-form solution, cf. Arkin and Evstigneev [1].

Our approach contributes to a positive theory of long-term dynamics in the sense that it provides a method to ensure that (non-)stationary impacts on an economy (for instance due to economic policy) do not cause unstable or chaotic dynamics. To achieve this goal we work within the theory of random dynamical systems, describing the stochastic law of motion and the exogenous stochastic perturbation by two coupled dynamical systems. The important concept for the study of the dynamics employed here is that of a random fixed point of a random dynamical system which is a stochastic analogue of a deterministic steady state. It is tailor-made for the study of the stability analysis of sample-paths in stochastic dynamic systems. A random fixed point is invariant under the stochastic law of motion where invariance is understood with respect to the exogenous perturbation. For each realization of the exogenous stochastic process, a random fixed point determines an entire sample-path with respect to which a definition of sample-path stability is given. Note that the well-known concept of Markov equilibria is not applicable to the class of models considered here. Beside the fact that we go beyond the Markov framework, a Markov equilibrium only captures the statistical properties of the long-run behavior and thus is not useful for the analysis of sample-path stability, see Schenk–Hoppé and Schmalfuss [15].

The applicability of our approach (which we believe to be mandatory for

[1] The approach presented in this paper therefore should provide a valuable tool in real business cycle theory where the "correctness" of filters is still an unsettled question, see e.g. Cogley and Nason [5].

any theory in economics) hinges on the availability of a constructive method to show existence and uniqueness of (globally) stable random fixed points. This method is provided by a Banach-type fixed point theorem for the class of models considered here, extending earlier work due to Schmalfuss [16,17]. The theorem gives sufficient conditions ensuring that the long-run behavior of all sample-paths is uniquely determined by the sample-path of a random fixed point. This, in particular, excludes sensitive dependence on initial conditions. Due to the fact that the result establishes the convergence property on a known subset of the state space it might become a valuable tool for studies based on numerical simulations. To show applicability of our approach, the concept of a random fixed point and the Banach theorem presented here are thoroughly explained and illustrated by two examples from stochastic growth theory.

The remainder of the paper is organized as follows. The next Section 2 introduces non-stationary random dynamical systems and explains how dynamic economic models can be (re)formulated to fit this framework. Then, in Section 3, random fixed points are defined and a Banach fixed point theorem is proved. Section 4 contains the examples.

2. Random Dynamical Systems with Non-Stationary Noise

This section provides an introduction to the mathematical framework used. It builds upon the theory of random dynamical systems which is comprehensively presented in the monograph by Arnold [2]. This theory offers a description of stochastic systems from a dynamical systems point of view and goes beyond the mere description of models with randomness by stochastic processes. Since stationarity of the underlying stochastic processes is usually assumed in the theory of random dynamical systems, we generalize this approach by including non-stationary stochastic processes here[2]. In particular the approach covers non-autonomous difference equations. In a related paper Kloeden et al. [10] consider non-autonomous random dynamical systems in connection with the numerical approximation of attractors.

After stating the general definition of a non-stationary random dynamical system, we thoroughly explain how to transform an economic model, which is described by a stochastic law of motion, to meet this definition.

Two technical remarks have to be made. First, in this paper we restrict ourselves to discrete-time systems with Euclidean state space $\mathbf{R}^d$. Most results, however, have a continuous-time analogue, and hold for more general state spaces. Second, we assume equalities and convergence properties to hold for all elements

[2] It is important to point out that in our framework, for instance, there is no equivalent concept of an invariant measure and the Multiplicative Ergodic Theorem is not available in general. One therefore has to resort to new techniques and more generally applicable tools which mostly are not developed at the date this paper is written. However, we partially close this gap here.

of a probability space and do not work with the more familiar "for almost all" assumption[3].

Definition 2.1. A (non-stationary, discrete-time) *Random Dynamical System* (RDS) with time $\mathbf{T} = \mathbf{Z}_+$ (one-sided time) or $\mathbf{T} = \mathbf{Z}$ (two-sided time) and state space $\mathbf{R}^d$ consists of two ingredients:
(i) a measurable dynamical system $(\Omega, \mathcal{F}, (\theta^t)_{t\in\mathbf{Z}})$, i.e. for all $t \in \mathbf{Z}$, $\theta^t : \Omega \to \Omega$ is a $\mathcal{F}, \mathcal{F}$-measurable map which satisfies the flow property

$$\theta^{t+s} = \theta^t \circ \theta^s \quad \text{for all } t, s \in \mathbf{Z}$$
$$\theta^0 = \mathrm{id}_\Omega,$$

and
(ii) a $\mathcal{B}(\mathbf{T}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d), \mathcal{B}(\mathbf{R}^d)$-measurable map

$$\varphi : \mathbf{T} \times \Omega \times \mathbf{R}^d \to \mathbf{R}^d$$

such that $\varphi(t, \omega) := \varphi(t, \omega, \cdot)$ satisfies the cocycle property

$$\varphi(t+s, \omega) = \varphi(t, \theta^s\omega) \circ \varphi(s, \omega) \quad \text{for all } t, s \in \mathbf{T} \text{ and all } \omega \in \Omega$$
$$\varphi(0, \omega) = \mathrm{id}_{\mathbf{R}^d} \quad \text{for all } \omega \in \Omega.$$

An RDS is called *continuous*, if $\varphi(t, \omega) : \mathbf{R}^d \to \mathbf{R}^d$ is a continuous function for all $t \in \mathbf{T}$ and all $\omega \in \Omega$.

The measurable dynamical system (i) is a model of the exogenous stochastic perturbation (or stochastic component of the model), and the cocycle (ii) describes the dynamics of the stochastic system which is coupled to the perturbation.

The time-one map $h(\omega) := \varphi(1, \omega)$ is called the *generator* of the RDS because, by the cocycle property, each map $\varphi(t, \omega)$ can be represented as a composition of the maps $h(\omega)$ and $h(\omega)^{-1}$. $\varphi(t, \omega)$ inherits the regularities (such as continuity or smoothness) of $h(\omega)$ for $t \geq 0$ and of $h(\omega)^{-1}$ for $t \leq 0$.

If time $\mathbf{T} = \mathbf{Z}_+$, the map $\varphi(t, \omega)$ is not necessarily invertible and thus we can study the dynamics of non-invertible random maps. Note that for $\mathbf{T} = \mathbf{Z}$ the cocycle property implies invertibility of $\varphi(t, \omega)$.

[3] In the applications we are interested in the noise consists of two components: one is a (deterministic) time-dependent function and the other is a stationary (or even ergodic) stochastic process. If the noise is stationary, it is possible to restrict the treatment to an invariant subset of the probability space of full measure. For instance the ergodic theorem holds in this sense, Arnold [2, Appendix A.1]. If the noise is a deterministic function, then a condition either holds at any point in time or the set of sample-paths (there is only one in this case) on which this condition holds is void. The main reason for not dealing with "for almost all" statements here is that in a non-stationary framework null-sets depend on time in general. Thus, invariant sets of full measure cannot be constructed which causes technical problems.

We next give an illustration of the applicability of the concept of an RDS in economic dynamics. We show that the above definition of an RDS covers in particular the following class of economic systems for which the evolution of the state is governed by a law of motion of the form

$$x_{t+1} = H(p(t), \xi_t(\omega), x_t) \tag{2.1}$$

with $H : \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^d \to \mathbf{R}^d$ a measurable map, $p : \mathbf{Z} \to \mathbf{R}^m$ a (deterministic) function, and $\xi_t : \Omega \to \mathbf{R}^n$, $t \in \mathbf{Z}$, a stochastic process (i.e. a sequence of random variables). $x_t \in \mathbf{R}^d$ is the state of the system at time t.

We assume that both the function p and the stochastic process ξ_t are exogenous to the economic system in the sense that they do not depend on the state of the system. From this point of view the law of motion is coupled to the noise process. The time-dependent function p causes the non-stationarity of the driving process even if ξ_t is stationary.

The interpretation we have in mind is as follows. For a constant deterministic function $p(t) \equiv c$, (2.1) describes the stochastic evolution of an economy over time. The law of motion is then given by $H_c(\cdot) := H(c, \cdot)$, $H_c : \mathbf{R}^n \times \mathbf{R}^d \to \mathbf{R}^d$ where the stochastic perturbation is exogenous. The function $p(t)$ models economic policy as a time-dependent intervention in the evolution of the economy. It is a time-variant change of the stochastic law of motion H_c.

For instance in the one-sector growth model with production shocks $\xi_t \in \mathbf{R}_+$ the law of motion is $x_{t+1} = H_c(\xi_t(\omega), x_t) := (1-\delta)x_t + \xi_t(\omega)\, f(x_t + c)$. $c \geq 0$ is a constant supply of foreign capital. Such models have been investigated in the case when ξ_t is a Markov processes, see Mirman [12,13]. If the supply of foreign capital is only temporary or time-variant (such as development aid), one has to resort to a model in which c is replaced by a time-dependent function $p(t)$. Analogously one can model export of capital or the temporary access to superior technologies.

We next show how an RDS is derived from (2.1). First, the two types of perturbations, time-dependent functions and stochastic processes, are modeled as measurable dynamical systems (we treat each case separately first and then put both models together into one dynamical system). Second, the generator h of the RDS φ is defined.

Modeling time-dependent functions as measurable dynamical systems. Given any function $p : \mathbf{Z} \to G$ with $(G, \mathcal{G})$ being a measurable space. Define $\Omega = \mathbf{Z}$, and let $\mathcal{F} = \mathcal{B}(\mathbf{Z})$ be the Borel σ-algebra on $\mathbf{Z}$. The maps $\theta^t : \Omega \to \Omega$ defined by $\theta^t(\omega) = t + \omega$ form a flow on Ω. θ^t denotes the t-th iterate of $\theta := \theta^1$. The tuple $(\Omega, \mathcal{F}, (\theta^t)_{t\in\mathbf{Z}})$ is a measurable dynamical system, and $p(t + \omega) = p(\theta^t\omega)$.

Modeling stochastic processes as measurable dynamical systems. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces. Given any family $\xi = (\xi_t)_{t\in\mathbf{Z}}$ of random variables $\xi_t : E \to G$, we can define the sample-path space $(\Omega, \mathcal{F}) = (G^{\mathbf{Z}}, \mathcal{G}^{\mathbf{Z}})$ (a measurable space) and the map $\theta^t : \Omega \to \Omega$, $\theta^t(\omega)(s) = \omega(t+s)$, for all $s \in \mathbf{Z}$, the left-shift on the space Ω. The family $(\theta^t)_{t\in\mathbf{Z}}$ forms a flow on Ω and θ^t denotes the t-th iterate of θ. The sample paths of the process ξ are reobtained by the evaluation map $t \mapsto \theta^t(\omega)(0) = \omega(t)$. The tuple $(\Omega, \mathcal{F}, (\theta^t)_{t\in\mathbf{Z}})$ is a measurable dynamical system.

Generation of φ by (2.1). Let $(\Omega_1, \mathcal{F}_1, (\theta_1^t)_{t\in\mathbf{Z}})$ and $(\Omega_2, \mathcal{F}_2, (\theta_2^t)_{t\in\mathbf{Z}})$ be measurable dynamical systems modeling the function $p(t)$ and the stochastic process ξ_t respectively. We define a new measurable dynamical system by $\Omega := \Omega_1 \times \Omega_2$, $\mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2$, and $\theta^t := (\theta_1^t, \theta_2^t)$. Further, let $h(\theta^t\omega, x) := H(p((\theta^t\omega)_1), (\theta^t\omega)_2(0), x)$. With these definitions (2.1) is equivalent to

$$x_{t+1} = h(\theta^t\omega, x_t). \tag{2.2}$$

The law of motion of the original system is thus represented as a function which is coupled to a dynamical system θ.

The solution $\varphi(t, \omega, x)$ of (2.2) satisfies for all $\omega \in \Omega$ and all $x \in \mathbf{R}^d$

$$\begin{aligned} &\varphi(t, \omega, x) = h(\theta^{t-1}\omega) \circ \ldots \circ h(\omega)\, x \quad \text{for all } t \geq 1, \\ &\varphi(0, \omega, x) = x \end{aligned}$$

and, if $h(\omega)$ is invertible,

$$\varphi(t, \omega, x) = h(\theta^t\omega)^{-1} \circ \ldots \circ h(\theta^{-1}\omega)^{-1}\, x \quad \text{for all } t \leq -1.$$

It is straightforward to check that (2.2) generates a random dynamical system with time $\mathbf{T} = \mathbf{Z}_+$. h is measurable because H, ξ, and p are measurable. If $h(\omega)$ is invertible and the inverse is measurable, then an RDS with time $\mathbf{T} = \mathbf{Z}$ is generated. Recall that the RDS with time $\mathbf{T} = \mathbf{Z}_+$ ($\mathbf{T} = \mathbf{Z}$) is continuous, if $h(\omega)$ is continuous (and, in addition, $h(\omega)^{-1}$ is continuous).

Thus far we have discussed the evolution of non-stationary stochastic systems on the level of sample-paths. Some remarks are in order on the distribution of the system at some fixed period in time.

Suppose that the deterministic part $p(t)$ is known for all times (for instance if $p(t)$ represents a given economic policy). No additional uncertainty is generated through its fluctuation under this assumption. Further denote by $\mathbf{Q}$ the probability measure on $(E, \mathcal{E})$ associated with the stochastic process ξ_t. Then $\mathbf{P} := \xi\mathbf{Q}$ is a probability measure on the sample-path space $(\Omega, \mathcal{F})$. Note that the process ξ is stationary if and only if $\mathbf{P}$ is invariant under the left-shift. In particular, if the process is i.i.d. with distribution ν, then $\mathbf{P} = \nu^{\mathbf{Z}}$ is the product measure.[4]

We are now in a position to derive the distribution of the stochastic system at time t. Given any initial state x and any initial time ω_1 (since for the deterministic

[4] Time-dependent functions can also be modeled in this fashion. For a function $p : \mathbf{Z} \to G$ we could have defined the sample-path space $(\Omega, \mathcal{F}) = (G^{\mathbf{Z}}, \mathcal{G}^{\mathbf{Z}})$ and the map $\theta^t : \Omega \to \Omega$, $\theta^t(\omega)(s) = \omega(t+s)$, $s \in \mathbf{Z}$. The measure on Ω is then given by $\mathbf{P} = \otimes_{t\in\mathbf{Z}} \mathbf{1}_{p(t)}$ which has the property that $\mathbf{P}\{t \mapsto p(t)\} = 1$. However, due to the technical difficulties pointed out in the previous footnote we do not employ this probabilistic model in this paper.

part $p((\theta^t\omega)_1) = p(\theta_1^t\omega_1) = p(t+\omega_1))$, the distribution of the random variable $\omega_2 \mapsto \varphi(t,(\omega_1,\omega_2),x)$ is given by the image measure $\varphi(t,(\omega_1,\cdot),x)\mathbf{P}$, where $\mathbf{P}$ is the distribution of the sample-path space of the stochastic part.

Suppose the component $p(t)$ is also unknown. Then the distribution of the random variable $\omega \mapsto \varphi(t,\omega,x)$ is given by the image measure $\varphi(t,\cdot,x)\nu \otimes \mathbf{P}$ where ν is any measure on $\mathbf{Z}$. For instance, let us interpret $p(t)$ as a time-dependent economic policy which becomes effective at a certain period in time. Then $\nu = p_a\delta_a + p_b\delta_b$ represents the belief that the policy becomes (resp. became, if a or b is negative) effective a resp. b time periods away from the current time t where probability p_a resp. $p_b = 1 - p_a$ is assigned to each single event.

3. Random Fixed Points and A Banach Fixed Point Theorem

This section contains the main tools for the stability analysis of sample-paths. We first introduce the concept of a fixed point for non-stationary random dynamical systems and then give a definition of sample-path stability. For the applicability of this concept it is mandatory to have a tool to ensure uniqueness and stability of random fixed points at our disposal. We therefore present a version of the Banach fixed point theorem for non-stationary random dynamical systems which was first proved by Schmalfuss [16,17] in the ergodic case. We apply this result in the next section, studying stochastic economic growth models. The results extend previous work which is mainly done in the ergodic case, see Arnold [2] and the references therein.

Here is the definition of the key concept used in all further considerations. Let $(\Omega, \mathcal{F}, (\theta^t)_{t\in\mathbf{Z}})$ be a measurable dynamical system and let φ be a random dynamical system with generator $h(\omega)$.

Definition 3.1. A *random fixed point* of a random dynamical system φ is a random variable $x^\star : \Omega \to \mathbf{R}^d$ such that

$$x^\star(\theta\omega) = h(\omega, x^\star(\omega)) \equiv \varphi(1,\omega,x^\star(\omega)) \text{ for all } \omega \in \Omega. \tag{3.1}$$

A random fixed point $x^\star$ is called *globally attracting* in a family of sets $(U(\omega))_{\omega\in\Omega}$, if for all $\omega \in \Omega$ and all $x \in U(\omega)$

$$\lim_{t\to\infty} \|\varphi(t,\omega,x) - x^\star(\theta^t\omega)\| = 0. \tag{3.2}$$

Equation (3.1) implies $x^\star(\theta^t\omega) = \varphi(t,\omega,x^\star(\omega))$ for all $t \in \mathbf{Z}$. Hence a random fixed point is a stochastic process which satisfies the random difference equation (2.1) and whose state is determined only by the dynamical system modeling the noise.

Stability of a random fixed point requires that for all $\omega \in \Omega$ the sample-path of all initial values in some set $U(\omega)$ converges to (and therefore eventually moves as) the sample-path $t \mapsto x^\star(\theta^t\omega)$ of the random fixed point. $x^\star(\omega)$ is the initial state corresponding to this sample-path.

An alternative way to characterize random fixed points is as follows. Define the skew-product flow $\Theta_t : \Omega \times \mathbf{R}^d \to \Omega \times \mathbf{R}^d$, $(\omega, x) \mapsto (\theta^t\omega, \varphi(t, \omega, x))$ for all $t \in \mathbf{T}$. Then $x^\star(\omega)$ is a random fixed point if and only if the graph of $x^\star$ is invariant under Θ_t.

We define two asymptotic properties of random variables which will be used to formulate the main result in this section.

Definition 3.2. (i) A random variable $g : \Omega \to \mathbf{R}^d$ is called *tempered*, if for all $\omega \in \Omega$ the sample path $t \mapsto g(\theta^t\omega)$ satisfies $\lim_{t\to\pm\infty} \frac{1}{|t|} \log^+ \|g(\theta^t\omega)\| = 0$. Equivalently, if for all $\omega \in \Omega$ the sample path $t \mapsto g(\theta^t\omega)$ grows sub-exponentially fast forward and backward in time, i.e. for all $\delta > 0$

$$\lim_{t\to\pm\infty} e^{-\delta|t|} \|g(\theta^t\omega)\| = 0. \tag{3.3}$$

(ii) A random variable *g fulfills the law of large numbers*, if for all $\omega \in \Omega$ one has existence of the limits

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t} g(\theta^i\omega) \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t} g(\theta^{-i}\omega). \tag{3.4}$$

In the main result of this paper on the existence of globally attracting random fixed points, stochastic processes that fulfill the law of large numbers play a central role. It is therefore important to point out that there are many non-stationary processes which have this property. For instance non-stationary processes with asymptotically mean stationary distribution, cf. Gray and Kiefer [9], fulfill the law of large numbers. Gray and Kiefer [9] give a detailed treatment of these processes as well as numerous examples. See also Kurz [11] who considers closely related non-stationary processes.

The following Lemma is a direct consequence of the above definition.

Lemma 3.3. For any $j \in \mathbf{Z}$,

(i) g is tempered if and only if $g \circ \theta^j$ is tempered; and

(ii) g fulfills the law of large numbers if and only if $g \circ \theta^j$ fulfills the law of large numbers. The limit (3.4) is the same for g and $g \circ \theta^j$.

The following example of one-dimensional affine difference equations with non-stationary noise illustrates the concept of a random fixed point. The results obtained will also be used in the section on economic growth.

Consider the law of motion

$$x_{t+1} = a(\theta^t\omega)x_t + b(\theta^t\omega), \tag{3.5}$$

with $x_t \in \mathbf{R}$. Let $a, b : \Omega \to \mathbf{R}$ be measurable maps.

For instance the cobweb model $p_{t+1} = p_t^{\alpha(\theta^t\omega)}\xi(\theta^t\omega)$ with stochastic processes $\alpha(\omega) > 0$, $\xi(\omega) > 0$ and state space $\mathbf{R}_{++}$ can be written in the form (3.5) by applying the transformation $x_t = \log p_t$.

The generator is given by the affine map $h(\omega, x) := a(\omega)x + b(\omega)$ and the corresponding RDS is continuous. If $a(\omega) \neq 0$ for all ω, then $h(\omega, x)$ is invertible (and the inverse is continuous) and generates a continuous RDS with two-sided time $\mathbf{T} = \mathbf{Z}$.

We have the following result

Lemma 3.4. Suppose $\log|a(\omega)|$ fulfills the law of large numbers and the limits satisfy

$$\lim_{t\to\infty} \frac{1}{t}\sum_{s=0}^{t} \log|a(\theta^s\omega)| < 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\sum_{s=0}^{t} \log|a(\theta^{-s}\omega)| < 0 \tag{3.6}$$

and $b(\omega)$ is tempered. Then

$$x^\star(\omega) = b(\theta^{-1}\omega) + \sum_{t=1}^{\infty}\prod_{s=1}^{t} a(\theta^{-s}\omega) \cdot b(\theta^{-(t+1)}\omega) \tag{3.7}$$

is the unique random fixed point of (3.5). $x^\star(\omega)$ is globally attracting on $\mathbf{R}$ and tempered. Moreover, it attracts any tempered random variable in the sense that $\lim_{t\to\infty} \|\varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega)) - x^\star(\omega)\| = 0$.

Proof. Let us first check that $x^\star$ fulfills the requirements of Definition 3.1. Measurability of $x^\star$ is obvious. The invariance property (3.1) is satisfied because

$$a(\omega)x^\star(\omega) + b(\omega) = b(\omega) + \sum_{t=0}^{\infty}\prod_{s=0}^{t} a(\theta^s\omega) \cdot b(\theta^{-(t+1)}\omega) = x^\star(\theta\omega)$$

for all $\omega \in \Omega$.

Existence of $x^\star(\omega)$ can be seen as follows.

$$|x^\star(\omega)| \leq |b(\theta^{-1}\omega)| + \sum_{t=1}^{\infty}\prod_{s=1}^{t} |a(\theta^{-s}\omega)| \cdot |b(\theta^{-(t+1)}\omega)|.$$

(3.6) ensures that for some $\varepsilon > 0$, $\prod_{s=0}^{t} |a(\theta^{-s}\omega)| < \exp(-\varepsilon\, t)$ for all sufficiently large t. Temperedness of b yields that for any $0 < \delta < \varepsilon$, $|b(\theta^{-t}\omega)| < \exp(\delta\, t)$ for all sufficiently large t. Putting these two observations together it is straightforward to see that $|x^\star(\omega)| < \infty$ for all $\omega \in \Omega$.

Analogous considerations show that $x^\star(\omega)$ is tempered.

To prove the attraction property, we use that

$$\varphi(t, \omega, x) = \prod_{s=0}^{t-1} a(\theta^s\omega) \cdot x + b(\theta^{t-1}\omega) + \sum_{i=1}^{t-1}\prod_{j=i}^{t-1} a(\theta^j\omega) \cdot b(\theta^{i-1}\omega) \tag{3.8}$$

for all $t > 1$ and therefore

$$|\varphi(t,\omega,x) - x^\star(\theta^t\omega)| = \prod_{s=0}^{t-1} a(\theta^s\omega) \cdot |x - x^\star(\omega)|.$$

The claim follows from assumption (3.6) which implies for a sufficiently small $\varepsilon > 0$ the estimate $\prod_{s=0}^{t-1} |a(\theta^s\omega)| < \exp(-\varepsilon\, t)$ for all large t.

Let us finally show that $\lim_{t\to\infty} \|\varphi(t,\theta^{-t}\omega, g(\theta^{-t}\omega)) - x^\star(\omega)\| = 0$ for any tempered random variable g. (3.8) implies

$$\varphi(t,\theta^{-t}\omega, g(\theta^{-t}\omega)) = \prod_{s=1}^{t} a(\theta^{-s}\omega) \cdot g(\theta^{-t}\omega) + b(\theta^{-1}\omega) + \sum_{i=1}^{t-1}\prod_{j=1}^{i} a(\theta^{-j}\omega) \cdot b(\theta^{i-1}\omega)$$

Since $\prod_{s=1}^{t} a(\theta^{-s}\omega) \cdot g(\theta^{-t}\omega) \to 0$ exponentially fast by assumption (3.6) and temperedness of g, we proved the assertion. □

For any family $(G(\omega))_{\omega\in\Omega}$ of subsets of $\mathbf{R}^d$ we define

$$\mathcal{G} := \{\text{all tempered random variables } g \text{ with } g(\omega) \in G(\omega) \text{ for all } \omega \in \Omega\}$$

Now we are in a position to state the version of the Banach fixed point theorem which is applicable to non-linear stochastic dynamic systems with non-stationary noise. The theorem generalizes previous results for the ergodic case which are due to Schmalfuss [16,17].

Theorem 3.5. Let φ be a continuous random dynamical system on $\mathbf{R}^d$ with time $\mathbf{T} = \mathbf{Z}_+$ over a measurable dynamical system $(\Omega, \mathcal{F}, (\theta_t)_{t\in\mathbf{Z}})$.

Suppose there exists a family $(G(\omega))_{\omega\in\Omega}$ of subsets of $\mathbf{R}^d$ such that $\mathcal{G}$ is non-empty and

(i) $(h(\theta^{-1}\omega, g(\theta^{-1}\omega)))_{\omega\in\Omega} \in \mathcal{G}$ for all $g \in \mathcal{G}$;

(ii) if, for some $g \in \mathcal{G}$, $(\varphi(t,\theta^{-t}\omega, g(\theta^{-t}\omega)))_{t\geq 0}$ is a Cauchy sequence for all $\omega \in \Omega$, then its limit is in $\mathcal{G}$; and

(iii) there exists a random variable $c(\omega)$ which fulfills the law of large numbers such that for all $\omega \in \Omega$

$$\sup_{x,y\in G(\omega), x\neq y} \log \frac{\|h(\omega,x) - h(\omega,y)\|}{\|x-y\|} \leq c(\omega) \tag{3.9}$$

and the limits fulfill

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=0}^{t} c(\theta^i\omega) < 0 \tag{3.10}$$

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=0}^{t} c(\theta^{-i}\omega) < 0. \tag{3.11}$$

Then there exists a random fixed point $g^\star \in \mathcal{G}$ which is unique in $\mathcal{G}$ and globally attracting in $(G(\omega))_{\omega\in\Omega}$, i.e.

(a) $g^\star(\theta\omega) = h(\omega, g^\star(\omega))$ for all $\omega \in \Omega$;

(b) any random variable in $\mathcal{G}$ which satisfies (a) is equal to $g^\star$; and

(c) for all $g \in \mathcal{G}$ and all $\omega \in \Omega$

$$\lim_{t\to\infty} \|\varphi(t, \omega, g(\omega)) - g^\star(\theta^t\omega)\| = 0$$

and

$$\lim_{t\to\infty} \|\varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega)) - g^\star(\omega)\| = 0$$

where in both cases the convergence is exponentially fast with rate given by (3.10) resp. (3.11).

If φ is continuously differentiable, then (3.9) becomes

$$\sup_{x\in G(\omega)} \log \left\| \frac{\partial\varphi(1, \omega, x)}{\partial x} \right\| \le c(\omega). \tag{3.12}$$

Conditions *(i)–(iii)* are invariance, completeness, and uniform average contraction assumptions, respectively. The above statement becomes a Banach fixed point theorem for continuous maps, if the noise process is trivial, i.e. if ω (and θ) can be suppressed in all expressions.

The invariance condition *(i)* requires that $h(\omega, G(\omega)) \subset G(\theta\omega)$ for all ω. In the example section 4 we will present growth models in which $G(\omega)$ is a deterministic set as well as a natural framework in which $G(\omega)$ cannot be chosen as a deterministic set.

Proof. The proof is a modified version of Schmalfuss [17, Proof of Theorem 2.2].

We first show that $\varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))$, $t \ge 0$, is a Cauchy sequence. We will use that (3.9) implies $\|h(\omega, x) - h(\omega, y)\| \le \exp(c(\omega))\,\|x - y\|$ for all $x, y \in G(\omega)$ and $\omega \in \Omega$. Define $\kappa(t, \omega) = \sum_{i=1}^{t} c(\theta^{-i}\omega)$ for $t \ge 1$ and $\kappa(0, \omega) = 1$.

Let $n > m \ge 1$. By the cocycle property and condition (iii), we obtain

$$\begin{aligned}
&\left\|\varphi(n, \theta^{-n}\omega, g(\theta^{-n}\omega)) - \varphi(m, \theta^{-m}\omega, g(\theta^{-m}\omega))\right\| \\
&= \left\|h(\theta^{-1}\omega) \circ \varphi(n{-}1, \theta^{-n}\omega, g(\theta^{-n}\omega)) - h(\theta^{-1}\omega) \circ \varphi(m{-}1, \theta^{-m}\omega, g(\theta^{-m}\omega))\right\| \\
&\le \exp(c(\theta^{-1}\omega))\ \left\|\varphi(n{-}1, \theta^{-n}\omega, g(\theta^{-n}\omega)) - \varphi(m{-}1, \theta^{-m}\omega, g(\theta^{-m}\omega))\right\| \\
&\le \exp(\kappa(m, \omega))\ \left\|\varphi(n{-}m, \theta^{-n}\omega, g(\theta^{-n}\omega)) - g(\theta^{-m}\omega)\right\|
\end{aligned}$$

An upper bound on the last term can be derived by adding the telescope sum $\pm\sum_{i=1}^{n-m-1} \varphi(i, \theta^{-i-m}\omega, g(\theta^{-i-m}\omega)) = 0$ and applying the same estimate as above. We find

$$\left\|\varphi(n{-}m, \theta^{-n}\omega, g(\theta^{-n}\omega)) - g(\theta^{-m}\omega)\right\|$$

$$\leq \sum_{i=0}^{n-m-1} \|\varphi(i+1, \theta^{-(i+m+1)}\omega, g(\theta^{-(i+m+1)}\omega)) - \varphi(i, \theta^{-(i+m)}\omega, g(\theta^{-(i+m)}\omega))\|$$
$$\leq \sum_{i=0}^{n-m-1} \exp(\kappa(i, \theta^{-m}\omega)) \, \|h(\theta^{-(i+m+1)}\omega, g(\theta^{-(i+m+1)}\omega)) - g(\theta^{-(i+m)}\omega)\|$$

Defining $\eta(\omega) = \|h(\theta^{-1}\omega, g(\theta^{-1}\omega)) - g(\omega)\|$, we get

$$\begin{aligned}\|\varphi(n, \theta^{-n}\omega, g(\theta^{-n}\omega)) - \varphi(m, \theta^{-m}\omega, g(\theta^{-m}\omega))\| & \\ &\leq \exp(\kappa(m,\omega)) \sum_{i=0}^{n-m-1} \exp(\kappa(i, \theta^{-m}\omega)) \, \eta(\theta^{-(i+m)}\omega) \\ &\leq \sum_{i=0}^{\infty} \exp(\kappa(i, \theta^{-m}\omega) + \kappa(m,\omega)) \, \eta(\theta^{-(i+m)}\omega)\end{aligned}$$

Note that $\kappa(i, \theta^{-m}\omega)) + \kappa(m,\omega) = \kappa(i+m,\omega)$. By condition (iii) one has that for any $\tilde{c}$ with $\lim_{t\to\infty} \frac{1}{t}\sum_{i=0}^{t} c(\theta^{-i}\omega) < \tilde{c} < 0$ there exists an $N_0(\tilde{c},\omega)$ such that $\kappa(m,\omega) < \tilde{c}\, m$ for all $m > N_0(\tilde{c},\omega)$. Further, by condition (i) and the definition of $\mathcal{G}$, $\eta(\omega)$ is tempered, i.e. for any $\delta > 0$ there exists an $N_1(\delta,\omega)$ such that $\eta(\theta^{-m}\omega) < \exp(\delta\, m)$ for all $m > N_1(\delta,\omega)$.

For any fixed $\tilde{c}$ from above we can chose a δ such that $\tilde{c}+\delta < 0$. We therefore obtain for all sufficiently large m that

$$\|\varphi(n, \theta^{-n}\omega, g(\theta^{-n}\omega)) - \varphi(m, \theta^{-m}\omega, g(\theta^{-m}\omega))\| \leq \sum_{i=0}^{\infty} \exp\left((\tilde{c}+\delta)\,(i+m)\right)$$

which implies that $\varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))$, $t \geq 0$, is a Cauchy sequence. By condition (ii), the limit $g^\star(\omega) := \lim_{t\to\infty} \varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))$ is in $G(\omega)$. The last equation also ensures that $g^\star$ is tempered and hence $g^\star \in \mathcal{G}$.

Further $g^\star(\omega)$ is a random fixed point, because continuity of h yields that for all $\omega \in \Omega$

$$\begin{aligned}h(\omega, g^\star(\omega)) &= h\big(\omega, \lim_{t\to\infty} \varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))\big) = \lim_{t\to\infty} h\big(\omega, \varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))\big) \\ &= \lim_{t\to\infty} \varphi\big(t+1, \theta^{-(t+1)}(\theta\omega), g(\theta^{-(t+1)}(\theta\omega))\big) = g^\star(\theta\omega)\end{aligned}$$

where the last equality is derived as above expect that one has to use Lemma 3.3 in addition.

Suppose there exist two random fixed points $g_1^\star$ and $g_2^\star$ in $\mathcal{G}$. Then, by the above,

$$\begin{aligned}\|g_1^\star(\omega) - g_2^\star(\omega)\| &= \|\varphi(t, \theta^{-t}\omega, g_1^\star(\theta^{-t}\omega)) - \varphi(t, \theta^{-t}\omega, g_2^\star(\theta^{-t}\omega))\| \\ &\leq \exp(\kappa(t,\omega)) \, \|g_1^\star(\theta^{-t}\omega) - g_2^\star(\theta^{-t}\omega)\|\end{aligned}$$

Since $g_1^\star$ and $g_2^\star$ are tempered, the last term tends to zero as t tends to infinity. This ensures assertion (b), i.e. uniqueness of the random fixed point in $\mathcal{G}$.

The exponentially fast convergence, (c), is immediate from the above procedure because

$$\begin{aligned}\|\varphi(t,\theta^{-t}\omega,g(\theta^{-t}\omega))-g^\star(\omega))\| &= \|\varphi(t,\theta^{-t}\omega,g(\theta^{-t}\omega))-\varphi(t,\theta^{-t}\omega,g^\star(\theta^{-t}\omega))\| \\ &\le \exp\left(\sum_{i=1}^{t} c(\theta^{-i}\omega)\right)\|g(\theta^{-t}\omega)-g^\star(\theta^{-t}\omega)\|\end{aligned}$$

and

$$\begin{aligned}&\|\varphi(t,\omega,g(\omega))-g^\star(\theta^t\omega))\| = \|\varphi(t,\omega,g(\omega))-\varphi(t,\omega,g^\star(\omega))\| \\ &\quad = \|h(\theta^{t-1}\omega)\circ\varphi(t-1,\omega,g(\omega))-h(\theta^{t-1}\omega)\circ\varphi(t-1,\omega,g^\star(\omega))\| \\ &\quad \le \exp\left(c(\theta^{t-1}\omega)\right)\|g(\omega)-g^\star(\omega)\| \le \exp\left(\sum_{i=0}^{t-1} c(\theta^{i}\omega)\right)\|g(\omega)-g^\star(\omega)\|\end{aligned}$$

□

4. Application to Stochastic Economic Growth

We present instructive examples to illustrate the applicability of the random fixed point concept and the Banach fixed point theorem. We intended to highlight that the class of tractable problems in stochastic economic growth is considerably extended by the method presented here. For instance, previous results due to Mirman [12,13] are generalized by allowing for regimes with unbounded growth and a non-stationary change of regimes. In the example the non-stationary component of the exogenous stochastic perturbation exhibits ongoing non-stationarity that does not vanish asymptotically. Examples of such processes are also provided by Gray and Kiefer [9] who study stochastic processes with asymptotically mean stationary distributions. While for the first class of examples the underlying family of sets $(G(\omega))_{\omega\in\Omega}$ consists of deterministic sets, the section closes with an illustration that one has, in general, to allow for stochastic sets.

4.1. Changing Regimes

We study a stochastic one-sector growth model with structural changes between a neoclassical and an AK regime. The regime-switching is modeled by a non-stationary process for which he non-stationarity does not vanish asymptotically. Moreover, the law of motion is stochastic in both neoclassical and AK regime, i.e. the law of capital accumulation is given by $k_{t+1} = f(\xi_t, k_t)$ in the neoclassical regime and by $k_{t+1} = A_t\, k_t$ in the AK regime. We assume for simplicity of presentation that the production shocks in either regime, ξ_t and A_t, are i.i.d. processes.

Denote by $R(t)$ the function which determines the current regime at time t. We assume that $R(t)$ is a periodic function with values in $\{1, 2\}$. Let the production shock $(\xi_t, A_t)_{t\in\mathbf{Z}}$ be an integrable i.i.d. process with values in a Borel-measurable set $Z_1 \times Z_2 \subset \mathbf{R}_{++} \times [1, \infty[$.

One has the equivalent representation of the processes over a common dynamical system $r(\omega) := R(\omega_1)$, $(\xi(\omega), A(\omega)) = \omega_2(0)$ where the corresponding dynamical system lives on the space $\Omega = \mathbf{Z} \times (Z_1 \times Z_2)^{\mathbf{Z}}$ with the flow being defined as $\theta^t : \Omega \to \Omega$, $\theta^t\omega = (t + \omega_1, \omega_2(t + \cdot))$.

We can and do restrict the probability space of the random variable (ξ, A) to an invariant subset of full measure on which the law of large numbers holds, cf. Definition 3.2 (ii), and on which A is tempered. The first restriction is possible by the ergodic theorem, Arnold [2, Appendix A.1]; and second restriction is allowed by integrability of A, see Arnold [2, Prop. 4.1.3 (ii)].

The stochastic law of motion is thus given by,

$$k_{t+1} = h(\theta^t\omega, k_t) \tag{4.1}$$

where

$$k_{t+1} = \begin{cases} f(\xi(\theta^t\omega), k_t) & \text{if } r(\theta^t\omega) = 1; \\ A(\theta^t\omega)\, k_t & \text{if } r(\theta^t\omega) = 2. \end{cases}$$

We assume that for all $\xi \in Z_1$, $f(\xi, \cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ is continuously differentiable, concave, and $\partial_k f(\xi, 0) > 1$. Further we suppose that $\xi \mapsto f(\xi, k)$ is increasing and there exists a $\underline{k} > 0$ such that for $\underline{\xi} := \inf Z_1$, $f(\underline{\xi}, k) > k$ for all $k \in]0, \underline{k}[$ and $f(\underline{\xi}, k) < k$ for all $k > \underline{k}$.

Denote by P the length of the period of the process r, and let P_i be the set of points in time during one period $t \in \{1, ..., P\}$ at which $r(\theta^t\omega) = i$. Define $\pi_i := |P_i|/P$ the fraction of time the regime i is observed.

We have the following result.

Corollary 4.1. If

$$\pi_1 \, \mathbf{E} \log f'(\xi(\omega), \underline{k}) + \pi_2 \, \mathbf{E} \log A(\omega) < 0 \tag{4.2}$$

then the conditions of Theorem 3.5 are satisfied and thus there exists a unique globally attracting (on $\mathbf{R}_{++}$) random fixed point for the stochastic growth model with regime-switching (4.1).

Proof. First note that $G = [\underline{k}, \infty[$ is invariant (in the sense that $h(\omega, G) \subset G$ for all ω) because $f(\omega, G) \subset G$ and $A(\omega) \geq 1$ for all ω. Moreover, any initial state of capital k is mapped into G after a finite time, if the production shock enters into f in a non-trivial way, because assumption (4.2) implies $\pi_1 > 0$. This result is also straightforward in the other stochastic cases. If the law of motion does not depend on the production shock and $A(\omega) \equiv 1$, then $\underline{k}$ is the unique asymptotically stable fixed point of the growth model on $\mathbf{R}_{++}$ and we are done.

The set $\mathcal{G}$ of all tempered random variables with values in G is non-empty because it contains all constant functions with values larger than $\underline{k}$. Let $g \in \mathcal{G}$

be any tempered random variable. $h(\theta^{-1}\omega, g(\theta^{-1}\omega)) \leq A(\theta^{-1}\omega)g(\theta^{-1}\omega)$ and temperedness of A implies condition (i) of Theorem 3.5.

Let us turn to the contraction condition (iii) of Theorem 3.5 before dealing with (ii).

Since $f(\xi, \cdot)$ is continuously differentiable, increasing, and concave, $f'(\xi, k)$ is a decreasing function which takes its maximum value on G at $\underline{k}$. Equation (3.9) in condition (iii) of Theorem 3.5 thus becomes $\log f'(\xi(\omega), \underline{k}) \leq c(\omega)$ for $r(\omega) = 1$ and $\log A(\omega) \leq c(\omega)$ for $r(\omega) = 2$.

Using the periodicity of the process r, we find that

$$\begin{aligned}
&\lim_{T\to\pm\infty} \frac{1}{|T|} \sum_{t=0}^{T} \log h'(\theta^t\omega, \underline{k}) \\
&= \lim_{T\to\pm\infty} \frac{1}{|T|\,P} \sum_{t=0}^{T} \Big(\sum_{s\in P_1} \log f'(\xi(\theta^t\omega), \underline{k}) + \sum_{s\in P_2} \log A(\theta^t\omega) \Big) \\
&= \frac{|P_1|}{P}\, \mathbf{E} \log f'(\xi(\omega), \underline{k}) + \frac{|P_2|}{P}\, \mathbf{E} \log A(\omega)
\end{aligned}$$

by the assumption that $(\xi(\theta^t\omega), A(\theta^t\omega))$ is an i.i.d. process. This observation implies that condition (iii) of Theorem 3.5 is satisfied under the assumption (4.2) of the Corollary.

It remains to consider condition (ii) of Theorem 3.5. The following analysis shows that for any random dynamical system which is generated by concave functions $k \mapsto h(\omega, k)$ and which possess a family of invariant sets $G(\omega) \equiv G = [\underline{k}, \infty[$ condition (iii) implies (ii).

Suppose $(\varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega)))_{t\geq 0}$ is a Cauchy sequence for all $\omega \in \Omega$ and $g \in \mathcal{G}$. The limit is in $G(\omega)$ because $G(\omega) \equiv G = [\underline{k}, \infty[$ is complete. It remains to show that the limit is tempered.

Since $k \mapsto h(\omega, k)$ is concave we have that

$$h(\omega, k) \leq h(\omega, \underline{k}) + h'(\omega, \underline{k})\, k \tag{4.3}$$

for all $k \geq 0$. The affine random dynamical system ψ generated by the law of motion $y_{t+1} = h(\theta^t\omega, \underline{k}) + h'(\theta^t\omega, \underline{k})\, y_t$ dominates the random dynamical system φ in the sense that $\psi(t, \omega, x) \geq \varphi(t, \omega, x)$ for all $t \geq 0$ and all ω. Therefore $\lim_{t\to\infty} \varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega)) \leq \lim_{t\to\infty} \psi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))$.

Lemma 3.4 ensures that under condition (iii) the random variable $y^\star(\omega) := \lim_{t\to\infty} \psi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))$ is the unique random fixed point of the random dynamical system ψ. Moreover $y^\star(\omega)$ is globally attracting on $\mathbf{R}$, tempered, and attracts any tempered random variable, i.e. $\lim_{t\to\infty} \|\psi(t, \theta^{-t}\omega, g(\theta^{-t}\omega)) - y^\star(\omega)\| = 0$. We thus obtain that $\lim_{t\to\infty} \varphi(t, \theta^{-t}\omega, g(\theta^{-t}\omega))$ is tempered. This finishes the proof. $\square$

To elaborate on condition (4.2), we give an example with small production shocks. Assume the production shocks enter the neoclassical production function such that $f'(z,k)$ depends continuously on z for any fixed $k > 0$. Fix any z. Then $f'(\xi, \underline{k}) < 1$ and thus $\mathbf{E} \log f'(\xi(\omega), \underline{k}) < 0$ if $\xi(\omega) \in \mathbf{Z}_1$ is in some small neighborhood of z. Hence any sufficiently small perturbation of a neoclassical regime in which the long-run behavior is described by a deterministic fixed point preserves the contraction condition (4.2) (with $\pi_1 = 1$) and thus Theorem 3.5 is applicable.

We can further permit occasional regime-switching. Given any process A such that $\mathbf{E} \log A(\omega) < \infty$, we find that for all sufficiently small $\pi_2 > 0$ condition (4.2) holds.

In particular, if $\xi(\omega) \equiv \xi$ and $A(\omega) \equiv A$ are constants, i.e. each regime is determined by a deterministic law of capital accumulation, then condition (4.2) follows from

$$\pi_1 \log f'(\xi, \underline{k}) + \pi_2 \log A < 0. \tag{4.4}$$

The nonempty set of admissible A is given by $A < f'(\xi, \underline{k})^{-\pi_1/\pi_2}$.

For instance, in the Cobb–Douglas case with $f(\xi, k) = (1-\delta)k + \xi k^\alpha$ one has $\underline{k} = (\delta/\xi)^{1/(\alpha-1)}$ and $f'(\xi, \underline{k}) = 1-(1-\alpha)\delta < 1$, if $\delta > 0$. Therefore condition (4.2) is satisfied for all $A < (1-(1-\alpha)\delta)^{-\pi_1/\pi_2}$.

The above growth model can be generalized in several ways. For instance, the regime-switching process can be replaced by a typical sample path of a process that is independent of the production shock. Moreover, any transient behavior is permitted, i.e. no restrictions have to be made on the regime-switching function in the short and medium run. Only in the long run the function has to show a statistically regular pattern that induces that time-averages take the form used in condition (4.2). One such example is e.g. the occurrence of the AK regime for T^α-times, $0 \le \alpha < 1$, during the time-span $\{0, ..., \pm T\}$ of length T. This regime-switching process also exhibits persistent non-stationarity. However, when calculating the time-averages, the contribution of the expansive AK regime to the contraction rate vanishes as time goes to infinity, because $T^\alpha/T \to 0$.

4.2. Recurrent Poverty

In the previous example the underlying family of sets $(G(\omega))_{\omega\in\Omega}$ could be defined by constructing one deterministic set. This feature is due to sufficiently high marginal returns in production close to the state of no capital.

In this section we study a stochastic neoclassical growth model in which marginal productivity can fall short of the rate of depreciation. Whenever this event occurs, the aggregate capital stock decreases and can get arbitrarily close to the state of no capital. This section thus illustrates that, in general, stochastic sets $(G(\omega))_{\omega\in\Omega}$ have to be allowed for.

We consider the stochastic law of motion

$$k_{t+1} = h(\theta^t\omega, k_t) := (1 - \delta(\theta^t\omega))\, k_t + f(k_t) \tag{4.5}$$

where f is a twice continuously differentiable neoclassical production function and $\delta(\omega) \in [0,1]$ is a stochastic rate of depreciation.

If no production is possible without capital, i.e. $f(0) = 0$, then any capital stock – no matter how small – decreases if $f'(0) < \delta(\theta^t\omega)$. Suppose runs of this event of arbitrary (finite) length occur along any sample-path of the stochastic rate of depreciation. Then any sample-path of the capital stock k_t gets arbitrarily close to zero. Therefore, the argument used in the preceding example does not apply here.

We next show how an invariant family of sets $(G(\omega))_{\omega\in\Omega}$, i.e. $h(\omega, G(\omega)) \subset G(\theta\omega)$ for all ω, can be constructed in such a case.

Since the open interval $]0, \infty[$ is invariant under the random dynamical system generated by (4.5), we find that

$$\begin{aligned}
\frac{1}{k_{t+1}} &= \frac{1}{[1 - \delta(\theta^t\omega) + f'(0)]\, k_t + f(k_t) - f'(0)\, k_t} \\
&= \frac{1}{[1 - \delta(\theta^t\omega) + f'(0)]\, k_t} + \frac{f'(0)\, k_t - f(k_t)}{[1 - \delta(\theta^t\omega) + f'(0)]\, k_t\, [1 - \delta(\theta^t\omega) + f(k_t)]} \\
&\le \frac{1}{[1 - \delta(\theta^t\omega) + f'(0)]\, k_t} + d
\end{aligned}$$

Finiteness of d is implied by our assumptions on the production function f.

The associated random difference equation, letting $y_t := 1/k_t$,

$$y_{t+1} = [1 - \delta(\theta^t\omega) + f'(0)]^{-1}\, y_t + d \tag{4.6}$$

is affine.

Suppose $1/[1 - \delta(\omega) + f'(0)]$ fulfills the law of large numbers and the corresponding limits are strictly negative. Then Lemma 3.4 applies and we obtain that the random dynamical system generated by (4.6), say ψ, has the unique globally attracting random fixed point

$$y^\star(\omega) = d \left(1 + \sum_{t=1}^{\infty} \prod_{s=1}^{t} [1 - \delta(\theta^{-s}\omega) + f'(0)]^{-1} \right)$$

Thus the set $[y^\star(\omega), \infty[$ is invariant for ψ.

Finally, using that ψ dominates φ in the sense that $\psi(t, \omega, 1/k) \ge \varphi(t, \omega, k)$, we find that

$$G(\omega) := \left[\frac{1}{y^\star(\omega)}, \infty \right[\tag{4.7}$$

is invariant for the random dynamical system φ.

The contraction condition *(iii)* of Theorem 3.5 is now obtained by inserting the left-hand side of (4.7) into (3.12).

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